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Research Article

Analytical Solution for Bending of Functionally Graded Timoshenko Nanobeams Incorporating Surface Energy and Microstructure Effects

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Abstract: The bending response of functionally graded material (FGM) Timoshenko nanobeams is investigated analytically. The material properties of the nanobeam are assumed to vary continuously through the thickness via power law function. In the context of the modified couple stress and surface elasticity theories, the governing equations and boundary conditions are derived using Hamilton's principle. The Navier solution is obtained for simply supported boundary condition, and exact formula is proposed for the deflection. Good agreement with literature is observed. Selected numerical results are presented to verify the efficiency of the present model.
Keywords: FGM Timoshenko nanobeam; linear bending; modified couple stress theory; surface elasticity theory; Navier solution.

INTRODUCTION

Functionally graded materials (FGMs) are manufactured by mixing two or more dissimilar materials, usually a metal and a ceramic. The properties of FG materials are varied continuously along a specific spatial direction, usually the thickness of a structure. FGMs can bear high temperature gradients; reduce thermal and residual stresses and other orthotropic mechanical properties, Udupa *et al.*, (2014). Therefore, FGMs have wide applications in engineering fields, such as micro/nanoelectromechanical systems. Studying the mechanical behavior of micro-/nanostructures made of FGMs has been highly regarded by researchers recently.

At micro/nanoscale, the classical continuum theories cannot predict correctly the experimentally detected size-dependent behavior of structures. To capture this behavior and overcome the drawback of classical continuum theories, more general higher-order continuum mechanics theories have been extended and employed such as modified couple stress theory (MCST). Based on MCST, researchers derived models to study the bending, vibration and buckling analyses of FG micro/nanobeams (Al-Basyouni, K. S. *et al.*, 2015; Şimşek, M. *et al.*, 2013) or sandwich nanobeams (Thai, H. T. *et al.*, 2015). Based on these microstructure-dependent models, it was concluded that the couple stress effect shows a stiffness enhancement effect (stiffness-hardening) for small–scale FG beams.

In nanoscale structures, there is an extreme increasing ratio of surface area–to–bulk volume, which is considered one of their most important characteristics. In spite of ignoring the surface energy effect in classical continuum mechanics, as it is small compared to the bulk energy, the surface energy cannot be neglected due to its significant contribution to the total energy.

In literature, few models have been developed to investigate the combined effects of couple stress and surface energy on the mechanical behavior of homogeneous nanobeams (Gao, X. L. 2015; Attia, M. A., & Mahmoud, F. F. 2016) and nanoplates (Shaat, M. *et al.*, 2014; Attia, M. A., & Mahmoud, F. F. 2017). Recently, Attia (2017) and Shanab *et al.*, (2017), each of them has developed nonclassical model for FGMs considering the combined effects of couple stress and surface energy on FGM nanobeams. The Navier solution procedure is a sufficient method used to determine the linear

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bending of nanobeam analytically (Reddy, J. N. 2011; Gao, X. L., & Mahmoud, F. F. 2014; Zhang, L. *et al.*, 2016; & Attia, M. A. 2017).

Governing Equations of Motion of Timoshenko Nanobeam

Consider an FGM nanobeam with a rectangular cross-section of length L, width b, and thickness h as depicted in Fig.1. The material at bottom surface ($z = -0.5h$) is pure ceramic and at top surface ($z = 0.5h$) is pure metal. Through the beam thickness, the proposed distributions of volume fractions of metal and ceramic follow power law (PL). Then, the effective local material property ($\mathcal{P}$) such as Young's modulus (E), Poisson's ratio (ν), mass density (ρ), microstructure material length scale (l) of the bulk, surface Lame's constants (λ^s and μ^s), surface residual stress (τ^s) and surface mass density (ρ^s) of a PL nanobeam can be described as follows:

$$\mathcal{P}(z) = \mathcal{P}_c + (\mathcal{P}_m - \mathcal{P}_c)\left(\frac{1}{2} + \frac{z}{h}\right)^k, \quad -\frac{h}{2} \leq z \leq \frac{h}{2} \tag{1}$$

where $\mathcal{P}_c$ and $\mathcal{P}_m$ are the corresponding material property at the lower (ceramic) and upper (metal) surfaces of the FG beam, respectively, and k refers to the gradient index which controls the material variation through the beam thickness.

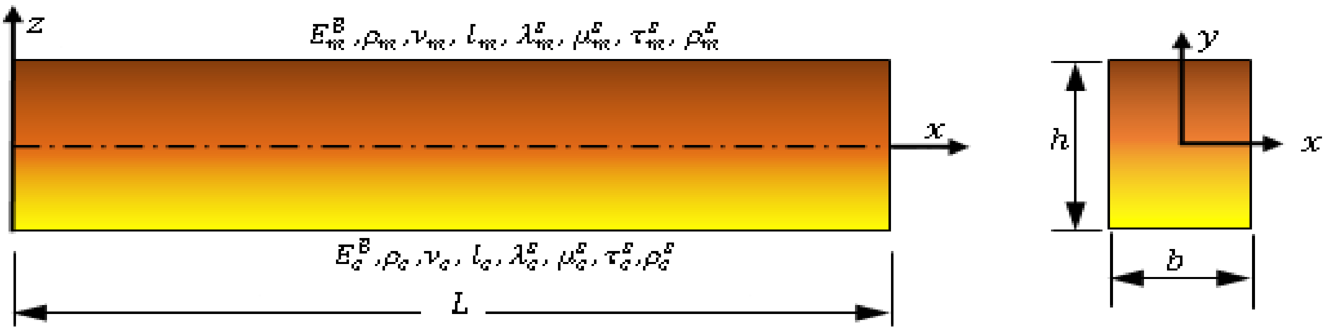

Fig. 1 Schematic of a functionally graded nanobeam with surface layer.

The governing equations of FGM Timoshenko nanobeam based on the combined effects of surface energy and microstructure is derived, in details, by Shanab *et al.*, (2017) incoperating von Kármán's geometrical nonlineairty. The linear model can be easily extracted as follows:

$$\mathrm{A}_{11}u'' - \mathrm{B}_{11}\psi'' = 0 \tag{2}$$

$$\left[\mathrm{S}_{11} + \mathcal{S}_1 + \frac{1}{2}\mathcal{C}_1\right]w'' - \frac{1}{4}S_{xy}w'''' - \frac{1}{4}S_{xy}\psi''' - \mathrm{S}_{11}\psi' + q + \frac{1}{2}\mathrm{f}_c' = 0 \tag{3}$$

$$-\mathrm{B}_{11}u'' + \frac{1}{4}S_{xy}w''' + \mathrm{S}_{11}w' + \mathrm{D}_{11}\psi'' - \mathrm{S}_{11}\psi - \frac{1}{2}\mathrm{f}_c = 0 \tag{4}$$

where u and w are the axial and lateral components, respectively, of an FGM Timoshenko beam. q is the z component of body force per unit length along the x −axis and f_c is the y −component of the body couple imposed on the sections as couple per unit axial length.

in which

$$\{\mathrm{D}_{11} \quad \mathrm{S}_{11}\} = \left\{\left(D_{xx} + \mathrm{I_P} + \frac{1}{4}S_{xy}\right) \quad \left(\mathcal{L}_2 - \frac{1}{2}\mathcal{L}_1 + k_s S_{xz}\right)\right\} \tag{5a}$$

$$\{\mathrm{A}_{11} \quad \mathrm{B}_{11}\} = \{(A_{xx} + \mathcal{C}_2) \quad (B_{xx} + \mathcal{P}_2)\} \tag{5b}$$

and

$$\{A_{xx} \quad B_{xx} \quad D_{xx}\} = \int_A \{1 \quad z \quad z^2\}[\lambda^B(z) + 2\mu^B(z)]dA,$$
$$\mu^B(z) = \frac{E(z)}{2(\nu(z)+1)} \quad \text{and} \quad \lambda^B(z) = \frac{E(z)\nu(z)}{(1+\nu(z))(1-2\nu(z))} \tag{6a}$$

$$\{S_{xz} \quad S_{xy}\} = \int_A \{1 \quad l(z)^2\}\mu^B(z)dA \tag{6b}$$

$$\{\mathcal{C}_1 \quad \mathcal{S}_1\} = \oint_{\partial A} \{1 \quad n_z^2\}\tau_s(z)dS \tag{6c}$$

$$\{\mathcal{C}_2 \quad \mathcal{P}_2 \quad \mathrm{I_P}\} = \oint_{\partial A} \{1 \quad z \quad z^2\}E_s(z)dS \tag{6d}$$

$$\{\mathcal{L}_1 \quad \mathcal{L}_2\} = \oint_{\partial A} n_y^2\{\tau_s(z) \quad \mu_s(z)\}dS \tag{6e}$$

Analytical Solution

In this section, analytical solutions for the linear bending of FGM Timoshenko nanobeam with simply supported boundary conditions are derived using the Navier solution. The displacement functions are expressed as product of undetermined coefficients and known trigonometric functions (shape functions) to satisfy the governing equations and the simply-supported (SS) boundary conditions at $x = 0, L$.

Neglecting the body couple f_c, the displacement field is assumed to be of the form

$$\begin{cases} w(x) = \sum_{k=1}^{N} W_n \sin \alpha x \\ u(x) = \sum_{k=1}^{N} U_n \cos \alpha x \\ \psi(x) = \sum_{k=1}^{N} \Psi_n \cos \alpha x \end{cases} \tag{7}$$

where (W_n, U_n, Ψ_n) are the Fourier coefficients to be determined for each n value, and $\alpha = \frac{n\pi}{L}$. The applied transverse load $q(x)$ is expanded in Fourier series as

$$q(x) = \sum_{k=1}^{N} Q_n \sin \alpha x \tag{8}$$

$$Q_n = \frac{2}{L}\int_0^L q(x) \sin \alpha x \; dx \tag{9}$$

The coefficients Q_n for some typical loads are:

$$Q_n = \begin{cases} \frac{4}{n\pi} q_0, n = 1,3,5, \dots & \text{uniform load of intensity } q_0 \\ q_0, n = 1 & \text{sinsoidal load of intensity } q_0 \end{cases} \tag{10}$$

Substituting Eqs. (7) and (8) into Eqs. (2-4) yields:

$$-A_{11}\alpha^2 U_n + B_{11}\alpha^2 \Psi_n = 0 \tag{11}$$

$$-\left(S_{11} + \mathcal{S}_1 + \frac{1}{2}\mathcal{C}_1\right)\alpha^2 W_n - \frac{1}{4}S_{xy}\alpha^4 W_n - \frac{1}{4}S_{xy}\alpha^3 \Psi_n + S_{11}\alpha \Psi_n + Q_n = 0 \tag{12}$$

$$B_{11}\alpha^2 U_n + S_{11}\alpha W_n - \frac{1}{4}S_{xy}\alpha^3 W_n - S_{11}\Psi_n - D_{11}\alpha^2 \Psi_n = 0 \tag{13}$$

Solving Eqs. (11-13), the coefficients (W_n, U_n, Ψ_n) are obtained as

$$W_n = \frac{1}{\Delta}\{\alpha^2[\alpha^2(A_{11}D_{11} - B_{11}^2) + A_{11}S_{11}]Q_n\} \tag{14}$$

$$U_n = \frac{1}{\Delta}\left\{\alpha^3\left[S_{11} - \frac{1}{4}S_{xy}\alpha^2\right]B_{11}Q_n\right\} \tag{15}$$

$$\Psi_n = \frac{1}{\Delta}\left\{\alpha^3\left[S_{11} - \frac{1}{4}S_{xy}\alpha^2\right]A_{11}Q_n\right\} \tag{16}$$

Where

$$\begin{aligned} \Delta = \frac{1}{4}S_{xy}\alpha^6[\alpha^2(A_{11}D_{11} - B_{11}^2) + 3A_{11}S_{11}] + \alpha^6 S_{11}[A_{11}D_{11} - B_{11}^2] \\ + \alpha^4\mathcal{S}_1[\alpha^2(A_{11}D_{11} - B_{11}^2) + A_{11}S_{11}] \\ + \frac{1}{2}\mathcal{C}_1\alpha^4[\alpha^2(A_{11}D_{11} - B_{11}^2) + A_{11}S_{11}] - \frac{1}{16}\alpha^8 A_{11}S_{xy}^2 \end{aligned} \tag{17}$$

Using Eqs. (14-16) in Eq. (6) yields the analytical expressions of $w(x), u(x)$ and $\psi(x)$ for the linear bending of an FGM Timoshenko nanobeam with simply supported boundary conditions incorporating the coupled effect of surface energy and microstructure.

Numerical Results

The obtained results using Navier solution for linear bending deflections are validated with the available literature. The material properties of FGM beam vary across the thickness according to a power law (PL). For the purpose of comparison, the dimensionless deflection was computed as $\overline{W} = 100w(EI_m/qL^4)$ for uniform load of intensity q. Tables 1 and 2 compare the dimensionless central deflection obtained using the present model for a simply supported homogeneous/FGM beams based on the CL (classical), CS (couple stress), SE (surface effect: residual and surface energy) and CSSER (couple stress-surface effect) analyses. As can be noticed, the comparisons in Tables 1–2 show that the present results are in a good agreement with those in the literature. In Table 2 and next results, the FG beam material properties are given in Table 3 (Hosseini-Hashemi and Nazemnezhad 2003)

Table 1 Dimensionless central deflection of a homogeneous Timoshenko beam under uniform load based on CS analysis, ($E = 1.44$ GPa, $\nu = 0.38$, $h = 88$ μm, $L = 20h$ and $b = 2h$).

	l/h					
	0.0	0.2	0.4	0.6	0.8	1.0
Present	1.3107	1.1166	0.7733	0.5118	0.3477	0.2465
Shanab *et al.*,. [4]	1.3107	1.1166	0.7733	0.5118	0.3477	0.2465
Reddy [3]	1.3103	1.1162	0.7731	0.5116	0.3475	0.2464

Table 2 Effect of the length-to-thickness ratio (L/h) on dimensionless central deflection of an FGM Timoshenko nanobeam, ($k = 1$, $h = 5$ nm, $b = 2h$, $l = h$).

L/h	Present				Shanab *et al.*,. [4]			
	CL	**SE**	**CS**	**CSSER**	**CL**	**SE**	**CS**	**CSSER**
5	1.8848	1.7026	0.3978	0.3889	1.8848	1.7026	0.3978	0.3889
7	1.7918	1.4892	0.3655	0.3509	1.7918	1.4892	0.3655	0.3509
10	1.7425	1.2375	0.3481	0.3219	1.7425	1.2375	0.3481	0.3219
15	1.7161	0.8979	0.3388	0.2872	1.7161	0.8979	0.3388	0.2872
25	1.7026	0.4825	0.3340	0.2236	1.7026	0.4825	0.3340	0.2236

Table 3 Material properties of FGM constituents.

Property	**Aluminum**	**Silicon**
Young's modulus (GPa)	$E_m = 70$	$E_c = 210$
Poisson's ratio	$\nu_m = 0.3$	$\nu_c = 0.24$
Residual surface stress (N/m)	$\tau_m^s = 0.9108$	$\tau_c^s = 0.6048$
Surface elastic modulus (N/m)	$E_m^s = 5.1882$	$E_c^s = -10.6543$

Table 4 Effect of FGM gradation index k on dimensionless central deflection, ($L/h = 15$, $b = 2h$, $l = h$).

k	$h = 5\ nm$				$h = 10\ nm$			
	CL	**SE**	**CS**	**CSSER**	**CL**	**SE**	**CS**	**CSSER**
0.3	2.0724	0.9780	0.4553	0.3656	2.0724	1.3291	0.4553	0.4056
1	1.7161	0.8979	0.3388	0.2872	1.7161	1.1791	0.3388	0.3108
2	1.5520	0.8566	0.2914	0.2529	1.5520	1.1041	0.2914	0.2708
3	1.4689	0.8340	0.2719	0.2383	1.4689	1.0640	0.2719	0.2540
10	1.2743	0.7741	0.2391	0.2133	1.2743	0.9633	0.2391	0.2254

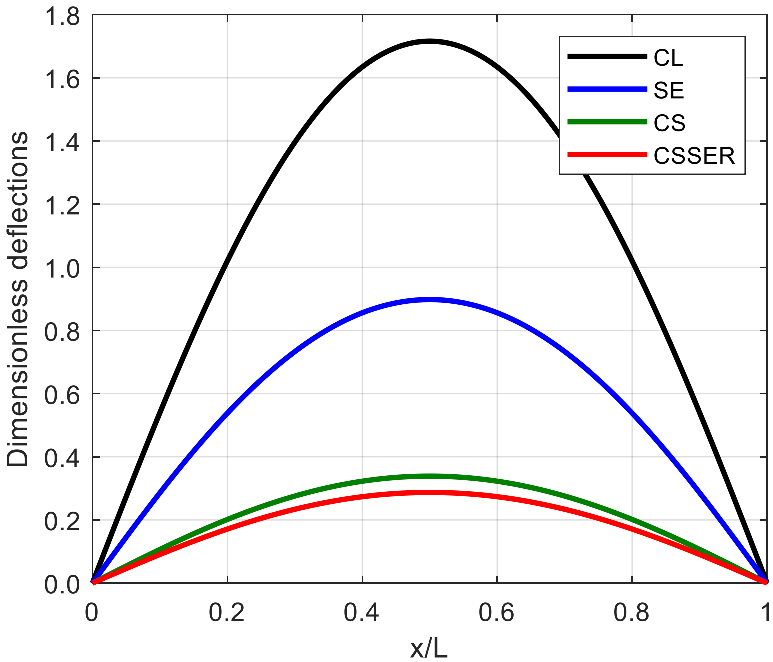

Fig. 2 Dimensionless deflections of simply supported FGM Timoshenko nanobeam for different analyses (CL, CS, SE and SEER) at ($k = 1, h = 5nm$, $b = h$, $L = 15h$ $l = h$).

As seen from Fig. 2, the nonclassical deflection; i.e. based on SE, CS and CSSER analyses, is less than the classical one. This because of the surface energy effect results in an increase in the beam stiffness. The couple stress effect also causes an increase in the beam rigidity (stiffness-hardening). Table 4 presents the analytical dimensionless central deflection according to various analyses (CL, SE, CS and CSSER) at different gradation index k for two values of the nanobeam thickness ($h = 5, 15$ nm). As observed from Table 4, the bending response based on CL and CS analyses are independent on the nanobeam thickness. It is also observed that as FGM index increases, the deflection decreases for all analyses. This can be interpreted since the stiffness of the nanobeam increases due to increase of ceramic constitution in the material of the beam. For a specific gradient index, SE and CS analyses show a deflection lower than the classical case. The combined effect of SE and CS (CSSER) has the lowest deflection among other analyses, as the stiffness in CSSER analysis becomes the greatest.

CONCLUSION

- Based on the Navier solution, an analytical formula for the linear bending response of FGM Timoshenko nanobeam is derived. Simultaneous effect of surface energy and microstructure is incorporated.
- Good agreement between the present results and those available in literature is obtained.
- The effects of different parameters such as FGM distribution index and beam thickness are investigated for both classical and nonclassical analyses (CL, CS, SE and CSEER).
- Comparing the effects of couple stress and surface energy, couple stress effect becomes the dominant at high values of h, whilst the surface energy is the dominant at very small thickness.
- Also, for all analyses, it can be concluded that under power law FGM model, increasing the gradient index value decreases the deflection.

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