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Review Article

Reducing Nonlinear Partial Differential Equation Using Lie Infinitesimals Method

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Abstract: Theoretical underpinnings of differential equations have advanced greatly, particularly in the twentieth century. This expansion is due to the rapid and effective development of supporting mathematical fields (such as functional analysis, measure theory, and function spaces), as well as an ever-increasing need for applications, particularly in engineering, science, and medicine. The Lie infinitesimals method was employed to reduce the nonlinear fourth order PDE into an ordinary differential equation then the resulting ODE solved by the finite difference method. The Lie infinitesimals method is used to solve significantly more complex problems which used in manufacture.
Keywords: Lie method, nonlinear, finite difference.

1. INTRODUCTION

In 1870, Lie discovered that all previous theories of ordinary differential equations integration can be reduced to a general form. In this way, it became possible to derive previous ideas from a common basis while also developing a larger perspective on differential equations theory in general. Adomian *et al*. [1] find different solutions of PDE. Yang *et al*. [2] find exact solutions of nonlinear PDE and also nonlinear transformations and reduction of nonlinear PDE to a quadrature. He *et al*. [3] find a new approach to nonlinear partial differential equations. Kirchheim *et al*. [4] Studied nonlinear PDE by geometry in matrix space. Rabinowitz *et al*.[5] used Some minimax theorems and applications to nonlinear partial differential equations. Rosinger *et al*. [6] Generalized a new solutions of nonlinear partial differential equations. Sirakov *et al*. [7] Solved uniformly elliptic fully nonlinear PDE. Liu *et al*.[8] Find a simple fast method in finding particular solutions of some nonlinear PDE [9-12]. find new methods to solve nonlinear PDE. Galaktionov *et al*. [13] find exact solutions and invariant subspaces of nonlinear partial differential equations in mechanics and physics. Odibat *et al*. [14] used numerical methods for nonlinear partial differential equations of fractional order. Reid *et al*. [15] Reduced of systems of nonlinear partial differential equations to simplified involutive forms. Sahadevan *et al*. [16] find exact solution of certain time fractional nonlinear partial differential equations [17-20] used Lie method to solve different type of PDE. The nonlinear fourth order PDE was reduced to an ordinary differential equation using the Lie infinitesimals approach, and the resulting ODE was solved using the finite difference method.

2. Lie infinitesimals method

Consider the generic example of a nonlinear differential equation system with p independent variables and q unknown functions.

$$\Delta^i\left(x, u_{(k)}\right) = 0, \qquad i = 1,2, \dots, m \tag{1}$$

The term u(k) is the kth derivative of u with respect to x, and m is the number of differential equations that characterise the system.

Consider a transformation with one parameter, α:

$$\bar{x} = \Xi(x, u; \alpha) \tag{2}$$

$$\bar{u} = \Theta(x, u; \alpha) \tag{3}$$

Where α is the transformation parameter? Assume that Ξ and Θ are sufficiently time-differentiable with respect to α. If ε is an infinitesimally small value of α, the expansion of the variables $\bar{x}$, $\bar{u}$ is defined by:

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$$\bar{x} = \Xi(x,u;0) + \varepsilon \frac{\partial \Xi}{\partial \alpha}(x,u;\alpha)|._{\alpha=0} + \frac{\varepsilon^2}{2!}\frac{\partial^2 \Xi}{\partial \alpha^2}(x,u;\alpha)|._{\alpha=0} + \cdots \quad (4)$$

$$\bar{u} = \Theta(x,u;0) + \varepsilon \frac{\partial \Theta}{\partial \alpha}(x,u;\alpha)|._{\alpha=0} + \frac{\varepsilon^2}{2!}\frac{\partial^2 \Theta}{\partial \alpha^2}(x,u;\alpha)|._{\alpha=0} + \cdots \quad (5)$$

These two equations could be simplified to:

$$\bar{x_i} = x_i + \varepsilon \xi_i(x,u) + O(\varepsilon^2), \quad i = 1,2,\dots,p \quad (6)$$

$$\bar{u}^a = u^a + \varepsilon \theta_a(x,u) + O(\varepsilon^2), \quad a = 1,2,\dots,q \quad (7)$$

Where ξ_i and θ_a are the infinitesimal transformations of independent and dependent variables, defined as:

$$\xi_i(x,u) = \frac{\partial \Xi}{\partial \alpha}(x,u;\alpha)|._{\alpha=0} \quad (8)$$

$$\theta_a(x,u) = \frac{\partial \Theta}{\partial \alpha}(x,u;\alpha)|._{\alpha=0} \quad (9)$$

The infinitesimal generator associated with (6) and (7) is given by the vector field:

$$\bar{V} \equiv \sum_{i=1}^{p} \xi_i(x,u)\frac{\partial}{\partial x_i} + \sum_{a=1}^{q} \theta_a(x,u)\frac{\partial}{\partial u_a} \quad (10)$$

The derivatives of u_a with regard to the independent variables are found in the field's prolongation. This can be summarised as follows:

$$Pr(\bar{V}) = \bar{V} + \sum_{a=1}^{q}\sum_{k}^{J} \theta_a^k\left(x, u_{(k)}\right)\frac{\partial}{\partial u_{a(k)}} \quad (11)$$

The similarity transformation invariance requirement is defined as follows:

$$\frac{\partial x_i}{\xi_i} = \frac{\partial u_j}{\theta_j}, \; i = 1,2,\dots,p, \quad j = 1,2,\dots,q \quad (12)$$

3. Fourth order nonlinear PDE (example)

Given the fourth order nonlinear PDE:

$$u_{tt} - u_{xx} - 2(u_x)^2 - 2uu_{xx} - u_{xxxx} = 0 \quad (13)$$

Now, assuming that Eq. (13) is invariant under the following one-parameter Lie group of transformations expressed as

$$\begin{aligned}
\hat{t} &= t + \varepsilon\tau(x,t,u) + O(\varepsilon)\\
\hat{x} &= x + \varepsilon\xi(x,t,u) + O(\varepsilon)\\
\hat{u} &= u + \varepsilon\eta(x,t,u) + O(\varepsilon)\\
\hat{u}_x^2 &= u_x^2 + \varepsilon\eta_x^2(x,t,u) + O(\varepsilon)\\
\hat{u}_{xx} &= u_{xx} + \varepsilon\eta^{xx}(x,t,u) + O(\varepsilon)\\
\hat{u}_{xxxx} &= u_{xxxx} + \varepsilon\eta^{xxxx}(x,t,u) + O(\varepsilon)
\end{aligned} \quad (14)$$

Where ε is the group parameter and ξ, τ, η are the infinitesimals and their corresponding extended infinitesimals of order 1, 2, and 4 are the functionsη^x, η_x^2, η^{xx}, η^{xxxx}, presented by

$$\begin{aligned}
\eta^x &= D_x(\eta) - u_x D_x(\xi) - u_t D_x(\tau)\\
\eta^{xx} &= D_x(\eta^x) - u_{xx} D_x(\xi) - u_{xt} D_x(\tau)\\
\eta^{xxxx} &= D_x(\eta^{xxx}) - u_{xxxx} D_x(\xi) - u_{xxxt} D_x(\tau)\\
\eta_x^2 &= D_x^2(\eta) + \tau D_x^2(u_t) - D_x^2(\tau u_t) + D_x^2(D_x(\xi)u) - D_t(\xi u) + \xi D_x^3(u)
\end{aligned} \quad (15)$$

Where D_x is the total derivative operator with respect to x written as?

$$D_x = \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + u_{xx}\frac{\partial}{\partial u_x} + u_{tx}\frac{\partial}{\partial u_t} \quad (16)$$

The generator of the one-parameter Lie group or the infinitesimal operator is the differential operator defined as

$$X = \tau(x,t,u)\frac{\partial}{\partial t} + \xi(x,t,u)\frac{\partial}{\partial x} + \eta(x,t,u)\frac{\partial}{\partial u} \quad (17)$$

The corresponding prolonged generator $P_r^{(4)}X$ of order (α, 4) is

$$P_r^{(4)}X = X + \eta_t^2 \frac{\partial}{\partial_t^2 u} + \eta^x \frac{\partial}{\partial u_x} + \eta^{xx} \frac{\partial}{\partial u_{xx}} + \eta^{xxx} \frac{\partial}{\partial u_{xxx}} + \eta^{xxxx} \frac{\partial}{\partial u_{xxxx}} \quad (18)$$

Applying the fourth prolongation $P_r^{(4)}X$ to the Eq. (13), we obtain the infinitesimal criterion of invariance corresponding Eq. (13), expressed as

$$\eta^{tt} - \eta^{xx} - 2\eta_x^2 - 2u\eta^{xx} - 2u_{xx}\eta - \eta^{xxxx} = 0 \quad (19)$$

Substituting the explicit expressions η^{xx},η^{tt}, η_x^2, and η^{xxxx} into (19) and equating powers of derivatives up to zero, we get an overdetermined system of linear partial differential equations; after resolving this system, the infinitesimals functions are given by

$$\tau(x,t,u) = C_1 t$$
$$\xi(x,t,u) = C_1 \frac{x}{2}$$
$$\eta(x,t,u) = -C_1 u - \frac{1}{2} C_2$$

Where C_1 is arbitrary constant? The corresponding Lie algebra is given by

$$X = C_1 t \frac{\partial}{\partial t} + C_1 \frac{x}{2}\frac{\partial}{\partial x} - C_1 u - \frac{1}{2} C_2 \frac{\partial}{\partial u} \quad (20)$$
$$X_1 = \frac{\partial}{\partial x},\ X_2 = t\frac{\partial}{\partial t} + \frac{x}{2}\frac{\partial}{\partial x} - u - \frac{1}{2}\frac{\partial}{\partial u} \quad (21)$$

Now, by solving the following characteristic equation

$$\frac{dt}{\tau(x,t,u)} = \frac{dx}{\xi(x,t,u)} = \frac{du}{\eta(x,t,u)}$$
$$\frac{dt}{t} = \frac{2dx}{x} = \frac{du}{-u - \frac{1}{2}}$$
$$\frac{dx}{x} = \frac{dt}{2t}$$
$$lnx = \frac{1}{2}\ln t + ln\eta_1$$
$$\eta_1 = \frac{x}{\sqrt{t}}$$
$$\frac{dt}{t} = \frac{du}{-u - \frac{1}{2}}$$
$$\ln t = -\ln\left(-u - \frac{1}{2}\right) + \ln f(\eta_1)$$
$$f(\eta_1) = -ut - \frac{1}{2}t$$
$$u = -f(\eta_1)t^{-1} - \frac{1}{2}$$

Apply chain's rule

$$u_t = -\frac{df}{d\eta_1}\frac{\partial\eta_1}{\partial t}t^{-1} + ft^{-2}$$
$$u_t = -f'\left(-\frac{1}{2}xt^{-\frac{3}{2}}\right)t^{-1} + ft^{-2}$$
$$u_{tt} = -\frac{d^2 f}{d\eta_1{}^2}\left(\frac{\partial\eta_1}{\partial t}\right)^2 t^{-1} - 2ft^{-3}$$
$$u_{tt} = -f''\left(\frac{1}{4}x^2 t^{-3}\right)t^{-1} - 2ft^{-3}$$
$$u_x = -\frac{df}{d\eta_1}\frac{\partial\eta_1}{\partial x}t^{-1} = -f't^{-\frac{3}{2}}$$
$$u_{xx} = -\frac{d^2 f}{d\eta_1{}^2}\left(\frac{\partial\eta_1}{\partial x}\right)^2 t^{-1} - \frac{df}{d\eta_1}\frac{\partial^2\eta_1}{\partial x^2}t^{-1} = -f''t^{-2}$$
$$u_{xxx} = -f'''t^{-\frac{5}{2}}$$

$$u_{xxxx} = -f^{(4)}t^{-3}$$

Substitute in eq. (13) we get:

$$-\frac{1}{4}\eta_1 f''t^{-3} - 2ft^{-3} + f''t^{-2} + 2f't^{-3} - 2ff''t^{-3} - f''t^{-2} + f^{(4)}t^{-3} = 0$$

$$-\frac{1}{4}\eta_1 f'' - 2f + 2f' - 2ff'' + f^{(4)} = 0 \qquad (22)$$

4. Finite Difference Method

Equation (22) can be rewrite as:

$$-0.25xy'' - 2y + 2y' - 2yy'' + y^{(4)} = 0 \qquad (23)$$

Using finite difference method

$$-0.25x_i \frac{y_{i-1}-2y_i+y_{i+1}}{h^2} - 2y_i + 2\frac{y_{i+1}-y_{i-1}}{2h} - 2y_i \frac{y_{i-1}-2y_i+y_{i+1}}{h^2} + \frac{y_{i-2}-4y_{i-1}+6y_i+4y_{i+1}+y_{i+2}}{h^4} = 0 \qquad (24)$$

$$y_{i-2} - (0.25x_ih^2 + h^3 + 4)y_{i-1} + (0.5x_ih^2 - 2h^4 + 6)y_i + (-0.25x_ih^2 + h^3 + 4)y_{i+1} + y_{i+2} - 2h^2y_iy_{i-1} + 4y_i^2h^2 - 2h^2y_iy_{i+1} = 0 \qquad (25)$$

$i = 3$

$$y_1 - (0.25x_3h^2 + h^3 + 4)y_2 + (0.5x_3h^2 - 2h^4 + 6)y_3 + (-0.25x_3h^2 + h^3 + 4)y_4 + y_5 - 2h^2y_3y_2 + 4y_3^2h^2 - 2h^2y_3y_4 = 0 \qquad (26)$$

$i = 4$

$$y_2 - (0.25x_4h^2 + h^3 + 4)y_3 + (0.5x_4h^2 - 2h^4 + 6)y_4 + (-0.25x_4h^2 + h^3 + 4)y_5 + y_6 - 2h^2y_4y_3 + 4y_4^2h^2 - 2h^2y_4y_5 = 0 \qquad (27)$$

$i = 5$

$$y_3 - (0.25x_5h^2 + h^3 + 4)y_4 + (0.5x_5h^2 - 2h^4 + 6)y_5 + (-0.25x_5h^2 + h^3 + 4)y_6 + y_7 - 2h^2y_5y_4 + 4y_5^2h^2 - 2h^2y_5y_6 = 0 \qquad (28)$$

$$F = \begin{bmatrix} 0.5x_3h^2 - 2h^4 + 6 & -0.25x_3h^2 + h^3 + 4 & 1 \\ -0.25x_4h^2 - h^3 - 4 & 0.5x_4h^2 - 2h^4 + 6 & -0.25x_4h^2 + h^3 + 4 \\ 1 & -0.25x_5h^2 - h^3 - 4 & 0.5x_5h^2 - 2h^4 + 6 \end{bmatrix} \begin{bmatrix} y_3 \\ y_4 \\ y_5 \end{bmatrix} + \begin{bmatrix} -2h^2y_3y_2 + 4y_3^2h^2 - 2h^2y_3y_4 \\ -2h^2y_4y_3 + 4y_4^2h^2 - 2h^2y_4y_5 \\ -2h^2y_5y_4 + 4y_5^2h^2 - 2h^2y_5y_6 \end{bmatrix} = \begin{bmatrix} -y_1 + (0.25x_3h^2 + h^3 + 4)y_2 \\ -y_2 - y_6 \\ -(-0.25x_5h^2 + h^3 + 4)y_6 - y_7 \end{bmatrix}$$

$$J = \begin{bmatrix} 0.5x_3h^2 - 2h^4 + 6 & -0.25x_3h^2 + h^3 + 4 & 1 \\ -0.25x_4h^2 - h^3 - 4 & 0.5x_4h^2 - 2h^4 + 6 & -0.25x_4h^2 + h^3 + 4 \\ 1 & -0.25x_5h^2 - h^3 - 4 & 0.5x_5h^2 - 2h^4 + 6 \end{bmatrix} + \begin{bmatrix} -2h^2y_2 + 8y_3h^2 - 2h^2y_4 & -2h^2y_3 & 0 \\ -2h^2y_4 & -2h^2y_3 + 8y_4h^2 - 2h^2y_5 & -2h^2y_4 \\ 0 & -2h^2y_5 & -2h^2y_4 + 8y_5h^2 - 2h^2y_6 \end{bmatrix}$$

Using numerical method (Newton Raphson method) we can find the final solution at N different values.

$$Y_{new} = Y_{old} - J^{-1} \qquad (29)$$

4. RESULTS AND DISCUSSION

Newton Raphson method gives us the final solution of equation (13) at different type of N based on the equation (29) and by using Matlab code.

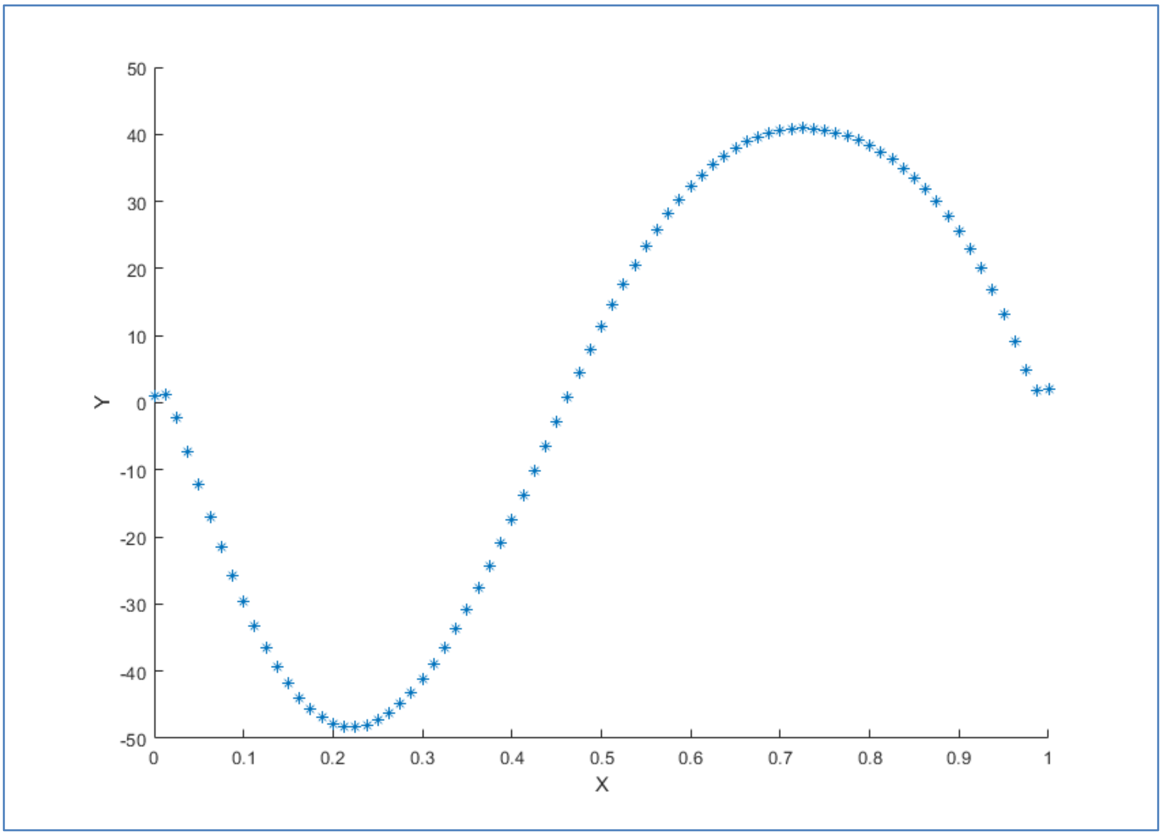

Fig-1: Solution of fourth order nonlinear PDE

5. CONCLUSIONS

Reducing of fourth order nonlinear PDE equation to ODE can be done by Lie infinitesimals method. Finite difference method can be used then to solve ordinary differential equation. Numerical method such Newton Raphson completes the problem to find solution in different values of N.

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