East African Scholars Journal of Engineering and Computer Sciences
Abbreviated Key Title: East African Scholars J Eng Comput Sci
ISSN 2617-4480 (Print) | ISSN 2663-0346 (Online) |
Published By East African Scholars Publisher, Kenya

OPEN ACCESS

Volume-2 | Issue-11 | Nov-2019 |
DOI:10.36349/EASJECS.2019.v02i11.009

Review Article

Study on the Relation between the Solution and Order of Fractional Bio-Heat Transfer Model

Zhoujin Cui
Jiangsu Maritime Institute, Jiangsu Nanjing 211170, China

*Corresponding Author
Zhoujin Cui

Abstract: This study focuses on analytical solution of a fractional Pennes bioheat transfer equation on skin tissue. The method of separating variables, finite Fourier sine transformation, Laplace transformation and their corresponding inverse transforms are used to solve this equation with three kinds of nonhomogeneous boundary conditions, namely, Dirichlet, Neumann and Robin boundary value conditions. The exact solutions are discussed and derived in the form of generalized Mittag-Leffler function. In addition, numerical results are presented graphically for various values of order factional derivative**.**

Keywords: Fractional order; Bio-heat transfer; Mittag-Leffler function; Separating variables; Finite Fourier sine transformation; Laplace transformation.

1. INTRODUCTION

Heat transfer is a very fundamental and important process in living things, especially in human bodies in order to maintain an almost constant temperature. Mathematical modeling of the complex thermal interaction between the vasculature and tissue has been a topic of interest for numerous physiologists, physicians, and engineers (Chato, J. C. 1981). The first quantitative relationship that described heat transfer in human tissue and included the effects of blood flow on tissue temperature on a continuum basis was presented by Harry H. Pennes (1948), a researcher at the College of Physicians and Surgeons of Columbia University. His landmark paper, which appeared in 1948, is cited in nearly all research articles involving bio-heat transfer. In an assessment by Eberhart *et al.*, it is concluded that this equation is "an adequate model for prediction of the macroscopic temperature distribution in several biological tissues" (Eberhart, R. C., Shitzer, A., & Hernandez, E. J. (1980). Later on, various researchers have reported studying of the bioheat transfer in the biological tissues. The most famous of them are Abramson (1967), Lemons *et al.*, (1987), Weinbaum *et al.*, (1984).

The past three decades have witnessed significant progress on fractional calculus. It has been found that fractional calculus is a mathematical tool that works adequately for anomalous social and physical systems with non-local, frequency- and history-dependent properties, and for intermediate states such as soft materials, which are neither ideal solid nor ideal fluid (Miller, K. S., & Ross, B.1993; Samko, S. G., *et al.*, 1993; Kiryakova. 1994; Yang, Z., *et al.*, 2012). Recently, numerical experiments have shown that in manyone-dimensional systems with total momentum conservation, the heat conduction does not obey the Fourier law (Khanna, F., & Matrasulov, D. (Eds.). (2006). Povstenko discussed a fractional heat conduction equation (Povstenko, Y. (2012). In this model the partial derivatives in classical diffusion equation is replaced by derivatives of non-integer order. Time-fractional heat conduction (or diffusion) was also studied in (Lenzi, E. K., *et al.*, 2009; Qi, H., & Liu, J. 2010; Yang, C., & Liu, F. 2006; Zheng, M., Liu, *et al.*, 2015; Zeng, F., *et al.*, 2015; Jiang, H., *et al.*, 2013; Zeng, F., *et al.*, 2013).

Quick Response Code

Journal homepage:
http://www.easpublisher.com/easjecs/

Article History
Received: 13.10.2019
Accepted: 26.10.2019
Published: 05.11.2019

Motivated by all the works above, we derive the analytical solution of the time-fractional Pennes bioheat transfer equation under three types of nonhomogeneous boundary conditions using the method of separating variables, finite Fourier sine transformation, Laplace transformation and their corresponding inverse transforms. The present paper considers the nonstochastic situation, and the solutions obtained here, in terms of the multivariate Mittag-Leffler function, contain corresponding results for boundary and initial value problems for the diffusion equation, namely, when $\alpha = 1$.

This paper is organized as follows. According to the fractional Fourier law, a fractional heat transfer equation is obtained in Section 2. The fractional heat transfer equation in the case $0 < \alpha \leq 1$ interpolates the standard heat conduction equation ($\alpha = 1$). In Section 2, we also give some relevant definitions and properties. In Section 3 we derive the analytical solution of the fractional Pennes bioheat transfer equation with Dirichlet and Neumann boundary conditions. Some conclusions are drawn in Section 4.

2. FORMULATION OF THE PROBLEM

The heat transfer in soft tissue during the thermal exposure to high temperature can be described using Pennes bio-heat equation, which is based on the classical Fourier law of heat conduction, $\mathbf{q} = -k\ \mathrm{grad}T$, in which k is the tissue thermal conductivity$(W/(m.K))$. It isconvenient to write the general form of Pennes bioheat equation asfollows:

$$\rho c \frac{\partial T}{\partial t} = -\nabla \mathbf{q} + W_b c_b (T_a - T) + Q_m,$$

where ρ, c are the density(kg/m^3), the specific heat of tissue$(J/(kg.K))$, c_b is the blood specific heat, T_a represents the temperature of arterial blood (K), W_b is themass flow rate of blood per unit volume of tissue$(kg/(s.m^3))$, Q_m is the metabolic heat generation per unit volume (W/m^3), T is the temperature rise above the ambient level.

In the theory of heat conduction proposed by Norwood (Norwood, F. R. 1972), the corresponding generalization of the Fourier law

$$\mathbf{q} = -k \int_0^t a(t-\tau)\mathrm{grad}(\tau) d\tau,$$

When kernel $a(t-\tau) = 1$ leads to the wave equation for the temperature and to the thermoelasticity without energy dissipation (Green, A. E., & Naghdi, P. M. 1993). The time-nonlocal dependence between the heat flux vector and the temperature gradient with the ``longtail" power kernel (Povstenko, Y. Z. 2008) can be interpreted in terms of fractional integrals and derivatives

$$\mathbf{q} = -\frac{k}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \frac{\partial}{\partial \tau} \mathrm{grad}(\tau) d\tau, \quad (0 < \alpha \leq 1). \qquad (2.1)$$

Considering The Definitions Of Caputo Fractional Derivative (Carstea D., & Carstea I. (2001)

$${}_0^c D_t^s u(x,t) = \frac{1}{\Gamma(n-s)} \int_0^t \frac{u^{(n)}(x,\tau)}{(t-\tau)^{s-n+1}} d\tau, \quad n = [s]+1,$$

Eq. (2.1) Becomes

$$\mathbf{q} = -k\, {}_0^c D_t^{1-\alpha} (\mathrm{grad}T), \quad (0 < \alpha \leq 1). \qquad (2.2)$$

We name Eq. (2.2) as the time fractional Fourier law. Eq. (2.2) converges to classical Fourier law for $\alpha \to 1$.

We can get the time-fractional bioheat transfer equation

$$\rho c \frac{\partial T}{\partial t} = k\, {}_0^c D_t^{1-\alpha} \nabla^2 T + W_b c_b (T_a - T) + Q_m, \quad (0 < \alpha \leq 1)$$

WHEN $\alpha = 1$, It Represents A Traditional Pennes' Bioheat Transfer Equation.

In the professional literature for the modelling of the human body and thermal comfort, there is a large variety of mathematical models on the heat transfer in different tissues of the human body. Skin is the more extensive living organ of the human body. It plays a variety of important roles including sensory, thermoregulation and host defense, etc. In this study, the time-fractional Pennes bioheat transfer equation with constant and transient heat flux conditions on the skin surface are studied analytically. Because of the geometrical features of the skin, the heat exchange in domain considered is assumed to be one-dimensional. The time-fractional Pennes equation for modeling skin tissue heat transfer is expressed as

$$\rho c\frac{\partial T}{\partial t} = k\, {}_0^cD_t^{1-\alpha}\frac{\partial^2 T}{\partial x^2} + W_b c_b(T_a - T) + Q_m. \qquad (2.3)$$

Where x is the distance from the skin surface

Besides the thermal parameters and metabolic rate of tissue, the skin temperature is also determined by any other factors such as the skin humidity, radiation emissivity of skin and parameters of surrounding air. These factors can be incorporated into the boundary condition (BC) at skin surface. The boundary condition for the heat transfer occurring at skin surface is generally included in one of the following kinds of conditions (Xu, F., *et al.*, 2008):

(1) Dirichlet condition (constant temperature): $T\mid_{skin} = T_\infty.(2.4)$
(2) Neumann condition (specified heat flux): $-k\frac{\partial T}{\partial n}\mid_{skin} = q_s.(2.5)$
(3) Convective condition: $-k\frac{\partial T}{\partial n}\mid_{skin} = h(T_\infty - T).(2.6)$
(4) Radiation condition: $-k\frac{\partial T}{\partial n}\mid_{skin} = \varepsilon\sigma h(T_\infty^4 - T^4).(2.7)$

Where is the outward normal at the boundary of computational domain, is skin emissivity and is Stefan-Boltzmann's constant in . First kind BC represents heating/cooling at a constant temperature , second kind BC represents heating/cooling by constant heat flux , third kind BC represents heating/cooling by convective heat transfer, which means heat exchange between the tissue surface and fluid at a constant temperature , and fourth BC represents heating/cooling by radiative heat transfer. Radiation is the loss of heat in the form of infrared waves. All objects continually radiate energy in accordance with the Stefan-Boltzmann law. When the surrounding is cooler than the body, net radiative heat loss occurs. Under normal conditions, close to half of body heat loss occurs by radiation. In contrast, a net heat gain via radiation occurs when the surrounding is hotter than the body.

3. ANALYTICAL SOLUTIONS

3.1. Dirichlet Condition.

In this part, in (2.3) is considered as zero. By the nature of fractional calculus (Miller, K. S., & Ross, B. (1993), if both sides of the Eq. (2.3) are multiplied by the operator, we can get the following equation

$$k\frac{\partial^2 T}{\partial x^2} = \rho c\, {}_0^cD_t^{\alpha} T + W_b c_b I_t^{\alpha-1}(T - T_a). \qquad (3.1)$$

We determine the solution of equation (3.1) with the initial condition

$$T(x,0) = \varphi(x), \quad 0 \le x \le L, \qquad (3.2)$$

And The Nonhomogeneous Dirichlet Boundary Conditions

$$T(0,t) = T_1, \quad T(L,t) = T_2. \qquad (3.3)$$

For Simplicity, Let $\theta = T - T_a$, Eq. (3.1) Becomes

$$k\frac{\partial^2 \theta}{\partial x^2} = \rho c\, {}_0^cD_t^{\alpha}\theta + W_b c_b I_t^{\alpha-1}\theta. \qquad (3.4)$$

The Initial and Boundary Conditions Can Be Written As

$$\theta(0,t) = T_1 - T_a, \quad \theta(L,t) = T_2 - T_a, \theta(x,0) = \varphi(x) - T_a, \quad 0 \le x \le L. \qquad (3.5)$$

Now taking finite Fourier Sine transform on (3.4) and (3.5), applying the Laplace transform, we can get

$$kn\pi\frac{(-1)^{n+1}(T_2 - T_a) + T_1 - T_a}{L}\frac{1}{s} - \frac{kn^2\pi^2}{L^2}\overline{\widetilde{\theta}} = \rho c s^{\alpha}\overline{\widetilde{\theta}} + W_b c_b s^{\alpha-1}\overline{\widetilde{\theta}} - \rho c s^{\alpha-1}\widetilde{\theta}(n,0), \qquad (3.6)$$in

WHICH $\bar{\bar{\theta}}$ IS THE LAPLACE TRANSFORM OF $\widetilde{\theta}$.
BY (3.6) WE HAVE

$$\bar{\bar{\theta}} = \frac{kn\pi\frac{(-1)^{n+1}(T_2-T_a)+T_1-T_a}{L}\frac{1}{s} + \rho c s^{\alpha-1}\widetilde{\theta}(n,0)}{\rho c s^{\alpha} + W_b c_b s^{\alpha-1} + \frac{kn^2\pi^2}{L^2}} = \Lambda_1 + \Lambda_2, \qquad (3.7)$$

Where

$$\Lambda_1 = \frac{kn\pi\frac{(-1)^{n+1}(T_2-T_a)+T_1-T_a}{L}}{s(\rho c s^{\alpha} + W_b c_b s^{\alpha-1} + \frac{kn^2\pi^2}{L^2})}, \quad \Lambda_2 = \frac{s^{\alpha}\rho c\widetilde{\theta}(n,0)}{s(\rho c s^{\alpha} + W_b c_b s^{\alpha-1} + \frac{kn^2\pi^2}{L^2})}. \qquad (3.8)$$

In order to avoid the burdensome calculations of residues and contour integrals, we apply the inverse Laplace transform method here.

First, We Rewrite (3.8) In A Series Form

$$\Lambda_1 = A\sum_{k=0}^{\infty}(-1)^k B^k \frac{s^{k-k\alpha-\alpha}}{(s+\frac{W_b c_b}{\rho c})^{k+1}}, \quad \Lambda_2 = \widetilde{\theta}(n,0)\sum_{k=0}^{\infty}(-1)^k B^k \frac{s^{k-k\alpha}}{(s+\frac{W_b c_b}{\rho c})^{k+1}}. \qquad (3.9)$$

In which

$$A = kn\pi\frac{(-1)^{n+1}(T_2-T_a)+T_1-T_a}{\rho c L}, \quad B = \frac{kn^2\pi^2}{\rho c L^2}.$$

Then, applying the inversion formulae term by term for the Laplace transform, (3.9) becomes

$$L^{-1}[\Lambda_2] = A\sum_{k=0}^{\infty}(-1)^k\frac{B^k}{k!}t^{\alpha+k\alpha}E^{(k)}_{1,1+k\alpha+\alpha-k}(-\frac{W_b c_b}{\rho c}t), \qquad (3.10)$$

$$L^{-1}[\Lambda_1] = \widetilde{\theta}(n,0)\sum_{k=0}^{\infty}(-1)^k\frac{B^k}{k!}t^{k\alpha}E^{(k)}_{1,1+k\alpha-k}(-\frac{W_b c_b}{\rho c}t), \qquad (3.11)$$

In Which

$$E_{\alpha,\beta}(t) = \sum_{k=0}^{\infty}\frac{t^k}{\Gamma(\alpha k+\beta)}, \quad E^{(k)}_{\alpha,\beta}(t) = \sum_{n=0}^{\infty}\frac{(n+k)!\,t^n}{n!\Gamma(\alpha n+\beta k+\beta)}$$

denote the generalized Mittag-Leffler function.

By the Fourier sine inverse transform, we obtain an exact solution as follows:

$$\theta(x,t) = \frac{2}{L}\sum_{n=1}^{\infty}\sum_{k=0}^{\infty}\widetilde{\theta}(n,t)\sin\frac{n\pi x}{L} \qquad (3.12)$$

Where

$$\widetilde{\theta}(n,t) = (-1)^k\frac{B^k}{k!}t^{k\alpha}\{At^{\alpha}E^{(k)}_{1,1+k\alpha+\alpha-k}(-\frac{W_b c_b}{\rho c}t) + \widetilde{\theta}(n,0)E^{(k)}_{1,1+k\alpha-k}(-\frac{W_b c_b}{\rho c}t)\}.$$

Thus we obtain an exact solution of (3.1)-(3.3) as follows

$$T(x,t) = T_a + \frac{2}{L}\sum_{n=1}^{\infty}\sum_{k=0}^{\infty}\widetilde{\theta}(n,t)\sin\frac{n\pi x}{L}. \qquad (3.13)$$

In addition to the above general solutions, the two different but simple heating situations are also demonstrated in the following cases.

When $\alpha = 1$, (3.9) Can Be Written As

$$\widetilde{\overline{\theta}} = \frac{kn\pi\frac{(-1)^{n+1}(T_2-T_a)+T_1-T_a}{L}\frac{1}{s} + \rho c\widetilde{\theta}(n,0)}{\rho cs + W_b c_b + \frac{kn^2\pi^2}{L^2}}.$$

Correspondingly, (3.13) Can Be Written As

$$\theta(x,t) = \frac{2}{L}\sum_{n=1}^{\infty}\widetilde{\theta}(n,t)\sin\frac{n\pi x}{L}$$

Where

$$\widetilde{\theta}(n,t) = \widetilde{\theta}(n,0)\exp(Bt - \frac{W_b c_b}{\rho c}t) + \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}(-1)^k A\frac{B^k t^{k+1}(-\frac{W_b c_b}{\rho c}t)^l (k+l)!}{m!\,l!\,(2k+l+1)!}.$$

In Particular, If, I.E., No Blood Perfusion Term, The Solution Of This Case Can Be Directly Obtained

$$T(x,t) = T_a + \frac{2}{L}\sum_{n=1}^{\infty}\widetilde{\theta}(n,0)\exp(Bt)\sin\frac{n\pi x}{L}$$

WHICH IS OBTAINED IN ANOTHER FORM IN (YANG, C., & LIU , F. 2006).

3.2. Neumann Condition.

The solution of equation (2.3) with the initial condition (3.2) and the nonhomogeneous Neumann boundary conditions (constant heat flux)

$$-k\frac{\partial T}{\partial x}\,|_{x=0} = q_1, \quad -k\frac{\partial T}{\partial x}\,|_{x=L} = q_2. \qquad (3.14)$$

In order to solve the problem with nonhomogeneous boundary, we firstly transform the nonhomogeneous boundary into a homogeneous boundary condition.

Let

$$T(x,t) = w(x,t) + v(x,t)$$

Where $w(x,t)$ Is A New Unknown Function And

$$v(x,t) = -\frac{q_1}{\mu}x + \frac{(q_1-q_2)x^2}{2\mu L}, \qquad (3.15)$$

Satisfies The Boundary Conditions

$$-\mu\frac{\partial v}{\partial x}\,|_{x=0} = q_1, \quad -\mu\frac{\partial v}{\partial x}\,|_{x=L} = q_2. \qquad (3.16)$$

The Function $w(x,t)$ Then Satisfies The Problem With Homogeneous Boundary Conditions

$$\begin{cases} \rho c\frac{\partial w}{\partial t} = k\,{}_0^c D_t^{1-\alpha}\frac{\partial^2 w}{\partial x^2} + f(x,t), \\ w(x,0) = \varphi_1(x), \quad 0 \le x \le L, \\ -\mu\frac{\partial w}{\partial x}\,|_{x=0} = 0, \quad -\mu\frac{\partial w}{\partial x}\,|_{x=L} = 0, \quad t \ge 0, \end{cases} \qquad (3.17)$$

In Which

$$\varphi_1(x) = \varphi(x) + \frac{q_1}{\mu}x - \frac{(q_1-q_2)x^2}{2\mu L}.$$

First consider the corresponding homogeneous equation of (3.17) (i.e. $f(x,t) = 0$).

Let $w(x,t) = X(x)T(t)$ and Substitute For $w(x,t)$ In (3.17), We Obtain An Ordinary Linear Differential Equation For $X(x)$

$$X''(x) + \frac{\rho c \lambda}{k} X(x) = 0, \quad X'(0) = X'(L) = 0, \tag{3.18}$$

Where The Parameter λ Is A Positive Constant The Sturm-Liouville Problem Given By (3.18) Has Eigenvalues

$$\lambda_n = \frac{n^2\pi^2 k}{\rho c L^2}, \quad n = 1, 2, \ldots, \tag{3.19}$$

And Corresponding Eigenfunctions

$$X_n(x) = \cos\frac{n\pi x}{L}, \quad n = 1, 2, \ldots. \tag{3.20}$$

Assume the Solution Of (3.20) Has The Following Form

$$w(x,t) = \sum_{n=1}^{\infty} P_n(t) \cos\frac{n\pi x}{L}. \tag{3.21}$$

In Order To Solve $P_n(t)$, We Expand $f(x,t)$ As A Fourier Cosine Series

$$f(x,t) = \sum_{n=1}^{\infty} f_n(t) \cos\frac{n\pi x}{L}, \tag{3.22}$$

In Which

$$f_n(t) = \frac{2}{L}\int_0^L f(x,t)\cos\frac{n\pi x}{L}dx, \; n = 1, 2, \ldots.$$

Substitute (3.21) and (3.22) Into (3.17), We Can Get

$$\rho c \frac{\partial P_n(t)}{\partial t} + \frac{n^2\pi^2 k}{L^2} {}_0^c D_t^{1-\alpha} P_n(t) = f_n(t). \tag{3.23}$$

By The Initial Condition In (3.17)

$$\sum_{n=1}^{\infty} P_n(0)\cos\frac{n\pi x}{L} = \varphi_1(x),\backslash$$

We Have

$$P_n(0) = \frac{2}{L}\int_0^L \varphi_1(x)\cos\frac{n\pi x}{L}dx, \; n = 1, 2, \ldots. \tag{3.24}$$

Applying the Laplace Transform On (3.23),

$$\rho c \; sL[P_n(t)] - P_n(0) + \frac{n^2\pi^2 k}{L^2} s^{1-\alpha} L[P_n(t)] - s^{-\alpha} P_n(0) = L[f_n(t)], \tag{3.25}$$

Thus

$$L[P_n(t)] = \frac{L[f_n(t)]}{\rho c s + \frac{n^2\pi^2 k}{L^2} s^{1-\alpha}} + \frac{P_n(0) + s^{-\alpha}P_n(0)}{\rho c s + \frac{n^2\pi^2 k}{L^2} s^{1-\alpha}}. \tag{3.26}$$

Applying the Inverse Laplace Transform Method Here, By

$$L^{-1}[\frac{1}{\rho c s + \frac{n^2\pi^2 k}{L^2} s^{1-\alpha}}] = \frac{1}{\rho c} L^{-1}[\frac{s^{\alpha-1}}{s^\alpha + \frac{n^2\pi^2 k}{\rho c L^2}}] = \frac{1}{\rho c} E_{\alpha,1}(-\frac{n^2\pi^2 k}{\rho c L^2} t^\alpha),$$

(3.26) Becomes

$$P_n(t) = \frac{1}{\rho c}\int_0^t E_{\alpha,1}(-\frac{n^2\pi^2 k}{\rho c L^2}\tau^\alpha)[f_n(t-\tau) + \frac{P_n(0)(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}]d\tau + P_n(0)E_{\alpha,1}(-\frac{n^2\pi^2 k}{\rho c L^2} t^\alpha). \tag{3.27}$$

Thus

$$w(x,t)=\sum_{n=1}^{\infty}\{\frac{1}{\rho c}\int_0^t E_{\alpha,1}(-\frac{n^2\pi^2k}{\rho cL^2}\tau^\alpha)[f_n(t-\tau)+\frac{P_n(0)(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}]d\tau$$
$$+P_n(0)E_{\alpha,1}(-\frac{n^2\pi^2k}{\rho cL^2}t^\alpha)\}\cos\frac{n\pi x}{L}. \tag{3.28}$$

By (3.15) and (3.28), We Obtain An Exact Solution As Follows

$$T(x,t)=\sum_{n=1}^{\infty}\{\frac{1}{\rho c}\int_0^t E_{\alpha,1}(-\frac{n^2\pi^2k}{\rho cL^2}\tau^\alpha)[f_n(t-\tau)+\frac{P_n(0)(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}]d\tau$$
$$+P_n(0)E_{\alpha,1}(-\frac{n^2\pi^2k}{\rho cL^2}t^\alpha)\}\cos\frac{n\pi x}{L}-\frac{q_1}{\mu}x+\frac{(q_1-q_2)x^2}{2\mu L}.$$

4. NUMERICAL COMPUTATION

Now we use the following parameter $L=0.15m$, $x=0.05m$, $k=0.5W/(m.K)$, $\rho=1000kg/m^3$, $c=3770J/(kg.K)$, $c_b=3800J/(kg.K)$, $W_b=0.5kg/(s.m^3)$, $\varphi(x)=37°C$, $T_a=37°C$, $T_1=37.5°C$,$T_2=37.8°C$.

We Select The Following Four Cases:

Case1. Time fractional equation with Dirichlet condition;

Case2. Time fractional equation with Dirichlet condition;

Case3. Time fractional equation with Dirichlet condition;

Case4. Standard equation with Dirichlet condition;

In these cases we find that as increases, growth trend of the temperature becomes slower. Temperature are shown graphically by Fig 1 for different values of.

5. CONCLUSION

We have derived the analytical solutions of the fractional Pennes bioheat transfer equation under three kinds of boundary conditions using the method of separating variables, the finite Fourier sine transformation, Laplace transformation and their corresponding inverse transforms. The time fractional derivative is considered in the Caputo sense. The solutions, which are given in the form of the generalized Mittag-Leffler function, reduce to those of the integer order equation and corresponding diffusion equations. In addition, numerical results are presented graphically for various values of order factional derivative.

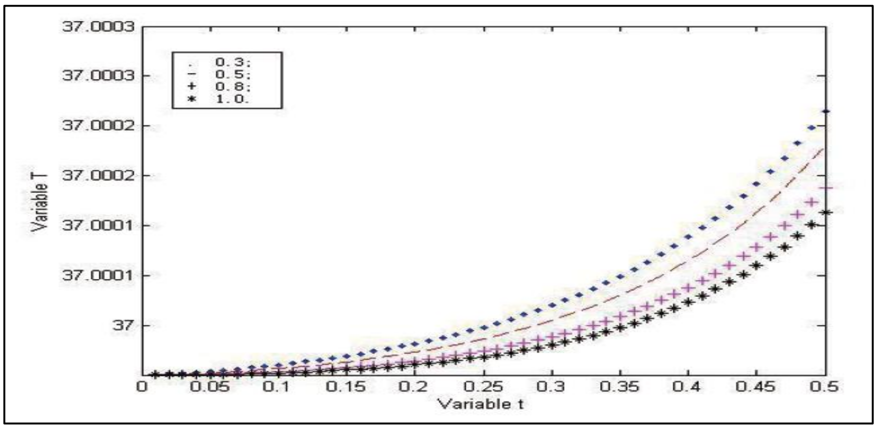

Fig.1. the behavior of $T(x,t)$ for different values of α when $x=0.05$

ACKNOWLEDGEMENT

The authors would like to express their gratitude to the referees for the comments and helpful suggestions.

REFERENCES

1. Chato, J. C. (1981). Reflections on the history of heat and mass transfer in bioengineering. *Journal of biomechanical engineering*, *103*(2), 97.
2. Pennes, H. H. (1948). Analysis of tissue and arterial blood temperatures in the resting human forearm. *Journal of applied physiology*, *1*(2), 93-122.
3. Eberhart, R. C., Shitzer, A., & Hernandez, E. J. (1980). Thermal dilution methods: estimation of tissue blood flow and metabolism. *Annals of the New York academy of sciences*, *335*(1), 107-132.
4. D.I, Abramson. (1967). Circulation in the Extremities, Academic press, New York.
5. Lemons, D. E., Chien, S., Crawshaw, L. I., Weinbaum, S., & Jiji, L. M. (1987). Significance of vessel size and type in vascular heat transfer. *American Journal of Physiology-Regulatory, Integrative and Comparative Physiology*, *253*(1), R128-R135.
6. Weinbaum, S. J. L. M., Jiji, L. M., & Lemons, D. E. (1984). Theory and experiment for the effect of vascular microstructure on surface tissue heat transfer—Part I: Anatomical foundation and model conceptualization.
7. Xu, F., Lu, T. J., & Seffen, K. A. (2008). Biothermomechanics of skin tissues. *Journal of the Mechanics and Physics of Solids*, *56*(5), 1852-1884.
8. K.S. Miller, B. Ross, An Introduction to the Fractional Calculus and Fractional Differential Equation, Wiley, NewYork, 1993.
9. Samko, S. G., Kilbas, A. A., & Marichev, O. I. (1993). *Fractional integrals and derivatives* (Vol. 1993). Yverdon-les-Bains, Switzerland: Gordon and Breach Science Publishers, Yverdon.
10. V. Kiryakova. (1994). Generalized Fractional Calculus and Applications, Pitman Research Notes Mathematical Series, vol. 301, Longman, Harlow.
11. Yang, Z., Cui, Z., & Zhang, R. (2012). Application of homotopy perturbation method to nonlinear fractional population dynamics models. *International Journal of Applied Mathematics and Computation*, *4*(4).
12. Khanna, F., & Matrasulov, D. (Eds.). (2006). *Non-linear dynamics and fundamental interactions* (Vol. 213). Springer Science & Business Media.
13. Povstenko, Y. (2012). Time-fractional radial heat conduction in a cylinder and associated thermal stresses. *Archive of Applied Mechanics*, *82*(3), 345-362.
14. Lenzi, E. K., da Silva, L. R., Silva, A. T., Evangelista, L. R., & Lenzi, M. K. (2009). Some results for a fractional diffusion equation with radial symmetry in a confined region. *Physica A: Statistical Mechanics and its Applications*, *388*(6), 806-810.
15. Qi, H., & Liu, J. (2010). Time-fractional radial diffusion in hollow geometries. *Meccanica*, *45*(4), 577-583.
16. Norwood, F. R. (1972). Transient thermal waves in the general theory of heat conduction with finite wave speeds.
17. Green, A. E., & Naghdi, P. M. (1993). Thermoelasticity without energy dissipation. *Journal of elasticity*, *31*(3), 189-208.
18. Povstenko, Y. Z. (2008). Fractional heat conduction equation and associated thermal stresses in an infinite solid with spherical cavity. *Quarterly journal of mechanics and applied mathematics*, *61*(4), 523-547.
19. Carstea D., & Carstea I. (2001). CAD in electrical engineering, Finiteelement method, Sitech, Romania.
20. Lin, Y., & Liu, F. (2008). Analytical solution for the non-homogeneous anomalous sub-diffusion equation. *Journal of Xiamen University (Natural Science)*, *47*(2), 158-163.
21. Yang, C., & Liu , F. (2006). A computationally effective Predictor-Corrector method for simulating fractional-order dynamical control system, *ANZIAM J., 47, (2006), 137-153.*
22. Zheng, M., Liu, F., Turner, I., & Anh, V. (2015). A novel high order space-time spectral method for the time fractional Fokker--Planck equation. *SIAM Journal on Scientific Computing*, *37*(2), A701-A724.
23. Zeng, F., Li, C., Liu, F., & Turner, I. (2015). Numerical algorithms for time-fractional subdiffusion equation with second-order accuracy. *SIAM Journal on Scientific Computing*, *37*(1), A55-A78.
24. Jiang, H., Liu, F., Meerschaert, M. M., & McGough, R. J. (2013). The fundamental solutions for multi-term modified power law wave equations in a finite domain. *Electronic journal of mathematical analysis and applications*, *1*(1), 55.
25. Zeng, F., Li, C., Liu, F., & Turner, I. (2013). The use of finite difference/element approaches for solving the time-fractional subdiffusion equation. *SIAM Journal on Scientific Computing*, *35*(6), A2976-A3000.

Financial Support:"This work was financially supported by the Major Projects of Natural Science Foundation of Colleges and Universities in Jiangsu Province（18KJ110003) and the Research Fund of Jiangsu Maritime Institute (No.2017KJYB-05)"