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**Original Research Article**

# Numerical Solution of Modified Burger Equation

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**Abstract:** In this paper, we present finite difference Algorithm called the Restrictive Taylor Approximation (RTA) is implemented to find the numerical solution of Burgers' equation which is nonlinear partial differential equation. This method is a new explicit method. The accuracy of the method is assessed in terms of the absolute error. Finally we derive the stability conditions in term of equation parameter and the discretization.
**Keywords:** Burgers' Equation, Restrictive Taylor Approximation, Finite difference, Exponential matrix

## INTRODUCTION

Burgers' equation was first introduced by Bateman (Bateman 1915) and then treated by Burger (Burger 1939; Burger 1948) as a mathematical model for turbulence. Burgers' equation can be solved exactly for a restricted set of initial functions. Hopf (Hopf 1950) and Cole (Cole 1951) independently showed that this equation can be transformed to a linear diffusion equation and solved exactly for arbitrary initial conditions.

Study of the general properties of Burgers' equation has motivated considerable attention due to its applications in areas such as number theory, gas dynamics, heat conduction and elasticity.

Many researchers have introduced various methods to solve Burgers' equation. For example, recent developments in this area include: invariantization of the Crank–Nicolson scheme for Burgers' equation developed by Kim (Kim 2008), a numerical method based on the Crank–Nicolson scheme developed by Kadalbajoo and Awasthi (Kadalbajoo and Awasthi 2006), Gülsu (Gülsu 2006) used finite difference approach to solve Burgers' equation.

Sakai and Kimura (Sakai and Kimura 2005) developed a numerical scheme based on a solution of nonlinear advection–diffusion equations, both Dag˘ and Saka (Dag˘ and Saka 2005) and Dhawan *et al*. (Dhawan *et al*. 2011) used cubic B-splines to develop a numerical method for Burgers' equation, Kutluay and Esen (Kutluay and Esen 2004) developed a linearized numerical scheme for Burgers'-like equations, Refik (Refik 2003) developed fully implicit finite-difference scheme for two-dimensional Burgers' equations, Kutluay (Kutluay *et al.*1999) developed explicit, Kapoor and Dhawan (Kapoor and Dhawan 2010) introduced Galerkin B-spline finite element method, finally exact-explicit finite difference methods for one-dimensional Burgers' equation and A restrictive type of Padé approximation (RPA)was introduced by Ismail ( Ismail *et al.* 2004).

Burger equation is solved numerically in all the previous references and the numerical solution is compared with the exact solution at various grid points. In this paper we introduce the numerical solution of both Burger and Modified Burger Equation by Restrictive Taylor Approximation.
Galerkin B-spline finite element method

## RESTRICTIVE TAYLOR APPROXIMATION FOR THE MODIFIED BURGER EQUATION

We use the Restrictive Taylor Approximation (RTA) as done in (Ismail *et al.* 2004; Ismail *et al.* 2001; Ismail *et al.* 2014). to solve the Modified Burger equation on the form:

$$\frac{\partial u}{\partial t} + u^2 \frac{\partial u}{\partial x} = \gamma \frac{\partial^2 u}{\partial x^2} \qquad a \le x \le b \quad t \ge 0 \tag{1}$$

where $a, b$ are constant.

We define first derivative with respect to *x*, *t* and the second derivative with respect to *x* on the form:

$$u_t|_{i,j} = \frac{u_{i,j+1} - u_{i,j}}{k} + O(k),$$
$$u_x|_{i,j} = \frac{u_{i+1,j} - u_{i-1,j}}{2h} + O(h^2) \qquad (2)$$
$$u_{xx}|_{i,j} = \frac{1}{h^2}\left[u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\right] + O(h^2)$$

where $u(x,t+k) = u(x,t) + k\frac{\partial u(x,t)}{\partial t} + \frac{k^2}{2!}\frac{\partial^2 u(x,t)}{\partial t^2} + \cdots$

$$= \left(1 + \frac{k}{1!}\frac{\partial}{\partial t} + \frac{k^2}{2!}\frac{\partial^2}{\partial t^2} + \cdots\right)u(x,t)$$

$$\mathrm{u_{i,j+1}} = Exp\left[\mathrm{k}\frac{\partial}{\partial \mathrm{t}}\right]\mathrm{u_{i,j}} \qquad (3)$$

$$u_{i,j+1} = Exp\left[-k\left(u_{i,j}D_x + D_x^2\right)\right]u_{i,j} \qquad (4)$$

The approximation of the partial derivative $D_x$ and $D_x^2$ at the grid point $(ih, jk)$ will take the forms

$$D_x u = \frac{u_{i+1,j} - u_{i-1,j}}{2h} \qquad (5)$$

$$D_x^2 u = \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2} \qquad (6)$$

where $\Delta x = h,\ \ \Delta t = k$

But restrictive Taylor's approximation of the first order $RT_{1,exp(xA)}$ of the exponential matrix function *exp(xA)*, as done in (Ismail *et al.* 2004; Ismail *et al.* 2001; Ismail *et al.* 2014), will take the form.

$$RT_{1,\exp(xA)} = I + x \in_{L_1} A \qquad (7)$$

where $A$ is $N-1\ x\ N-1$ real constant matrix, $I$ is the identity matrix and $\in_{L_1} = [\in_{i,L_1}]$ is the diagonal matrix of the restrictive term.

Then the restrictive Taylor's approximation for the Modified Burger's equation in scalar form is:

$$u_{i,j+1} = \frac{k}{h^2}\in_{i,L_1}\left(\gamma + u^2\frac{h}{2}\right)u_{i-1,j} + \left(1 - 2\frac{k}{h^2}\in_{i,L_1}\gamma\right)u_{i,j} + \frac{k}{h^2}\in_{i,L_1}\left(\gamma - u^2\frac{h}{2}\right)u_{i+1,j} \qquad (8)$$

## NUMERICAL EXAMPLE FOR RESTRICTIVE TAYLOR APPROXIMATION FOR MODIFIED BURGER'S EQUATION

**Example 1:**

The modified Burger's equation

$$\frac{\partial u}{\partial t} + u^2\frac{\partial u}{\partial x} = \gamma\frac{\partial^2 u}{\partial x^2} \qquad 0 \le x \le 1 \quad t \ge 0 \qquad (9)$$

with initial condition

$$u(x,0) = \frac{x}{a}\left[\frac{1}{1 + 2\sqrt{a}\ \ e^{\frac{x^2}{4\gamma a}}}\right] \qquad (10)$$

and the boundary conditions

$u(0,t) = 0 \qquad t \ge 0$

$$u(1,t) = \frac{1}{t+a}\left[\frac{1}{1 + 2\sqrt{t+a}\ \ e^{\frac{1}{4\gamma(t+a)}}}\right] \qquad t \ge 0 \qquad (11)$$

which have the exact solution on the form.

$$u(x,t) = \frac{x}{t+a}\left[\frac{1}{1+2\sqrt{t+a}\ \ e^{\frac{x^2}{4\gamma(t+a)}}}\right] \qquad (12)$$

To find the numerical solution, it was divided the position to $h = 0.1$ and the time to $k = 0.001$ and solve example (1) using the finite difference equation (8), the absolute error in **Table 1** and **Table 2** at $\gamma = 0.1,1$ respectively for various values of time and position.

**Table 1 : The absolute error of the solution of example (1) using *RT* method at time step *k=0.001* and *h=0.1* for various values of x,t and $\gamma = 0.1$**

| T | Absolute Error | | |
|---|---|---|---|
| | *x* = 0.2 | *x* = 0.6 | *x* = 0.8 |
| **0.01** | $1.289974832507923 \times 10^{-7}$ | 0.000001194264601350747 | 0.000005008138047699062 |
| **0.05** | 0.000002512991102633643 | 0.00003168231523245635 | 0.00012874527839969174 |
| **0.1** | 0.000006169142558735963 | 0.00012622392097404178 | 0.0004851849648204454 |
| **0.5** | 0.00019828444675275975 | 0.002851451452167003 | 0.007390814397592371 |
| **1** | 0.001349697294548688 | 0.008793049315289847 | 0.017611803433968537 |
| **2** | 0.005045758840212829 | 0.020037672598708124 | 0.031830557034620344 |
| **5** | 0.010692658655127588 | 0.027242762369861888 | 0.038764400922676044 |
| **10** | 0.010578993018306978 | 0.03133530370052398 | 0.04108530866726362 |

For *γ=1* the executive time of calculating 10 000 steps by restrictive Taylor method is 3.203 Seconds which is relatively very small.

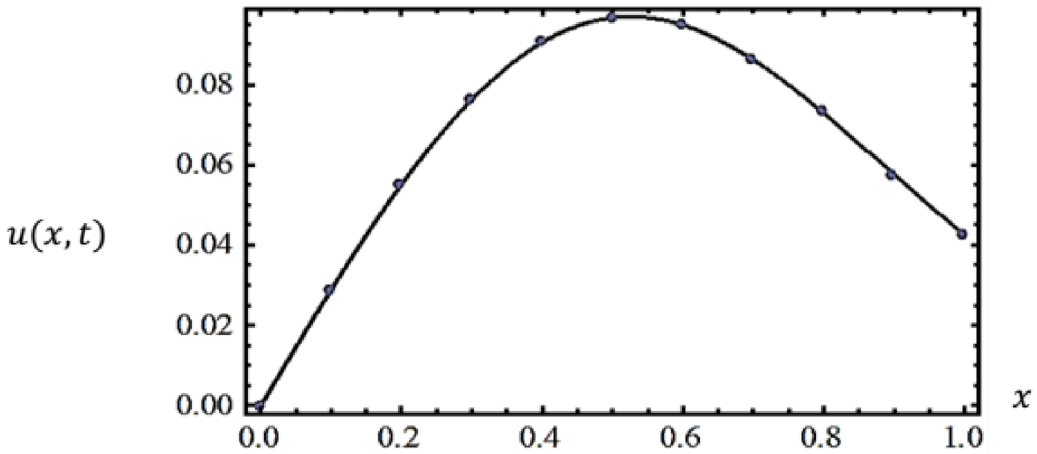


**Figure 1: Exact and Numerical RT Solution of Example (1) at *t=0.1* and $\gamma = 0.1$**

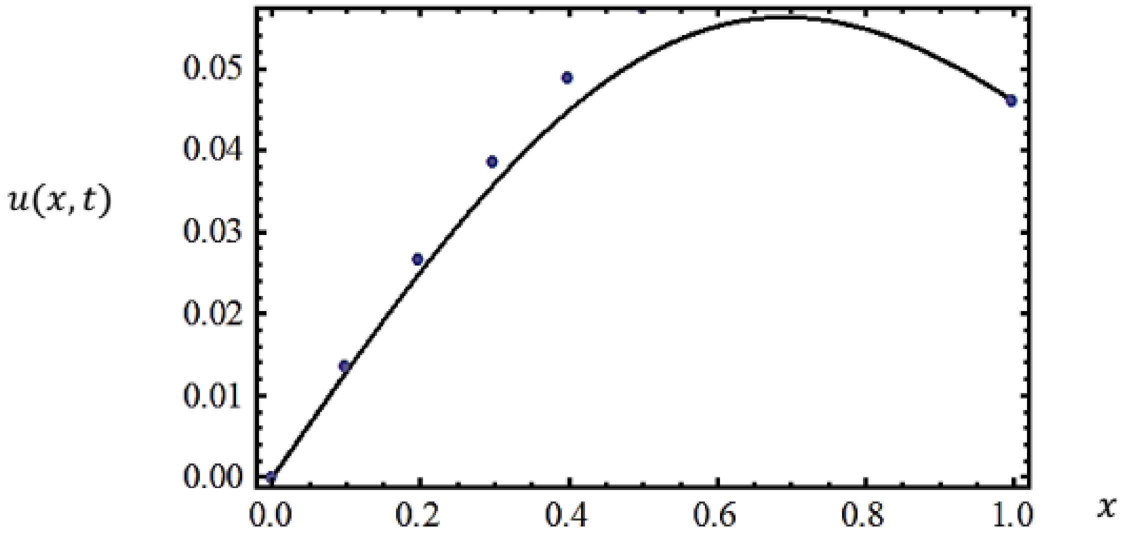


**Figure 2: Exact and Numerical RT Solution of Example 1 at *t=1* and *γ=0.1***

The above graph represents that both numerical and exact solutions are very closed.

**Table 2: The absolute error of the solution of example (1) using RT method at time step *k=0.001* and *h=0.1* for various values of *x,t* and *γ=1***

| T | Absolute Error | | |
|---|---|---|---|
| | $x$ = 0.2 | $x$ = 0.6 | $x$ = 0.8 |
| **0.01** | $2.745265777759353 \times 10^{-7}$ | $1.484122550965416 \times 10^{-7}$ | $1.484122550965416 \times 10^{-7}$ |
| **0.05** | 0.000004235099682128873 | 0.000003150560674097802 | 0.000008524815594795898 |
| **0.1** | 0.000009753654956638769 | 0.000007253406597629297 | 0.000018268965774115298 |
| **0.9** | 0.00005195857731206399 | 0.00005379952961603873 | 0.000016025375920186713 |
| **1** | 0.000052432983182786747 | 0.00005776412993115099 | 0.00002030735695178476 |
| **2** | 0.000022215489171761166 | 0.000047179975946443076 | 0.000014909349841801811 |
| **5** | 0.000015206053958995505 | 0.00002473842201736534 | 0.00001555695379497346 |
| **10** | 0.000004817937606033454 | 0.000008247235845958503 | 0.000005442065277598898 |

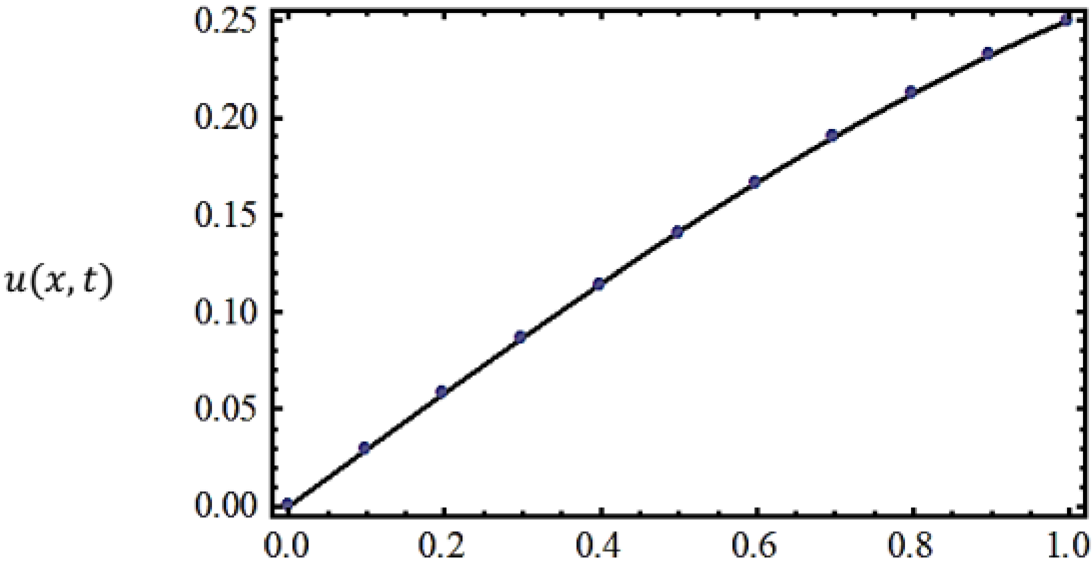


**Figure 3: Exact and Numerical RT Solution of Example 1 *at t=0.1* and *γ=1***

For *γ=1* the executive time of calculating 10 000 steps by restrictive Taylor method is 3.07 Seconds which is relatively very small.

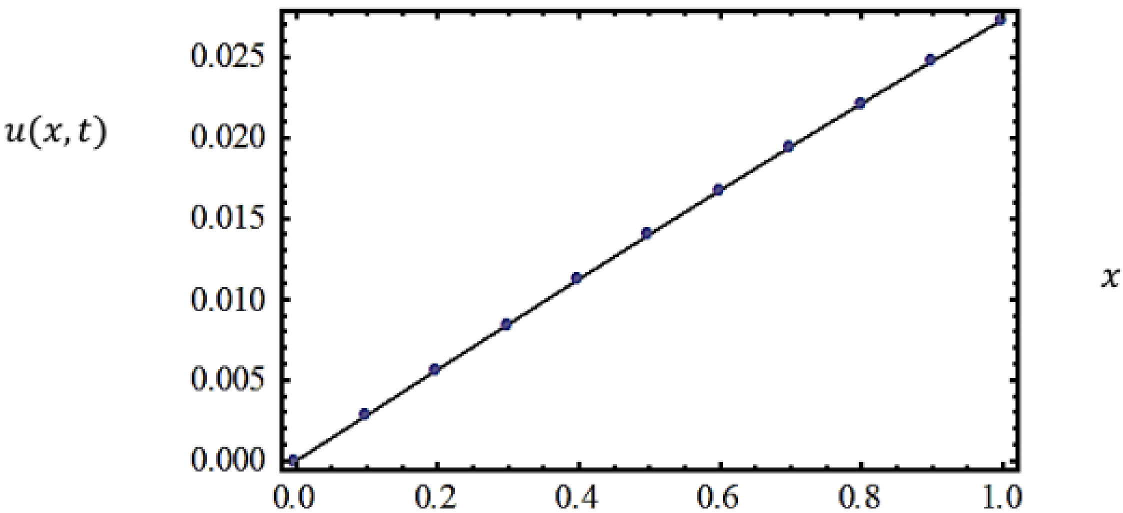


**Figure 4: Exact and Numerical RT Solution of Example 1 at *t=5* and *γ=1***

**Example 2:**

Restrictive Approximation for Viscous Burger Equation on the form:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \alpha\frac{\partial^2 u}{\partial x^2} \qquad a \le x \le b \quad t \ge 0 \tag{13}$$

It contains the simplest form of nonlinear advection term $uu_x$ and dissipation term $\nu u_{xx}$ where $\alpha = 1/Re$ ($\alpha$: kinematics viscosity and *Re*: Reynolds number) for simulating the physical phenomena of wave motion and thus determines the behavior of the solution.

At $0.5 \le x \le 1.5 \quad t \ge 0$ the exact solution is given by

$$u(x,t) = \frac{\alpha}{1+\alpha t}\left[x + \tan\frac{x}{2(1+\alpha t)}\right] \tag{14}$$

The boundary and the initial conditions are taken from the exact solution.
with initial condition $u(x,0) = \alpha x + \tan\frac{x}{2}$
and the boundary conditions $u(0.5,t) = \frac{\alpha}{1+\alpha t}\left[0.5 + \tan\frac{1}{(1+\alpha t)}\right] \quad t \geq 0$

$$u(1.5,t) = \frac{\alpha}{1+\alpha t}\left[2 + \tan\frac{1}{(1+\alpha t)}\right] \quad t \geq 0$$

Then the restrictive Taylor's approximation for the Burger's equation is:

$$u_{i,j+1} = \frac{k}{h^2} \in_{i,L} \left(\alpha + u\frac{h}{2}\right) u_{i-1,j} + \left(1 - 2\frac{k}{h^2} \in_{i,L} \alpha\right) u_{i,j} + \frac{k}{h^2} \in_{i,L} \left(\alpha - u\frac{h}{2}\right) u_{i+1,j} \quad (15)$$

The computational domain is [0.5,1.5] at $\alpha = 0.0001$. The numerical results are obtained by RTA scheme Eq. (15) at $h = 0.2, k = 0.001$. In Table 3, we show the absolute error. We expand the computation domain to [-10,10] and plot the RTA solution and exact solution in Fig. 7 for $\alpha = 0.0001$ with $h = 0.4$ and $k = 0.001$ at times *t =1.*

**Table 3: The absolute error of example (2) using RT method at time step k=0.001 and h=0.2 for various values of (x, t)**

| t | Absolute Error | | |
|---|---|---|---|
| | ***x* = 0.4** | ***x* = 1** | ***x* = 1.6** |
| **0.05** | $6.16639985601130 \times 10^{-19}$ | $1.34441069388202 \times 10^{-17}$ | $1.4154259361798 \times 10^{-16}$ |
| **0.09** | $2.71728169479179 \times 10^{-18}$ | $4.48588648865877 \times 10^{-17}$ | $4.6414695004104 \times 10^{-16}$ |
| **0.1** | $3.51688079699985 \times 10^{-18}$ | $5.55653613398821 \times 10^{-17}$ | $5.7386820989657 \times 10^{-16}$ |
| **0.5** | $1.08196600550475 \times 10^{-16}$ | $1.42811110159790 \times 10^{-15}$ | $1.4465045914541 \times 10^{-14}$ |
| **1** | $4.43106651631247 \times 10^{-16}$ | $5.72933085522808 \times 10^{-15}$ | $5.7918622155261 \times 10^{-14}$ |
| **2** | $1.79373795547790 \times 10^{-15}$ | $2.29492260698377 \times 10^{-14}$ | $2.317348786820 \times 10^{-13}$ |
| **5** | $1.12993788587908 \times 10^{-14}$ | $1.43506680063548 \times 10^{-13}$ | $1.4474244285772 \times 10^{-12}$ |
| **10** | $4.53665899876991 \times 10^{-14}$ | $5.73795188880260 \times 10^{-13}$ | $5.7804613417915 \times 10^{-12}$ |

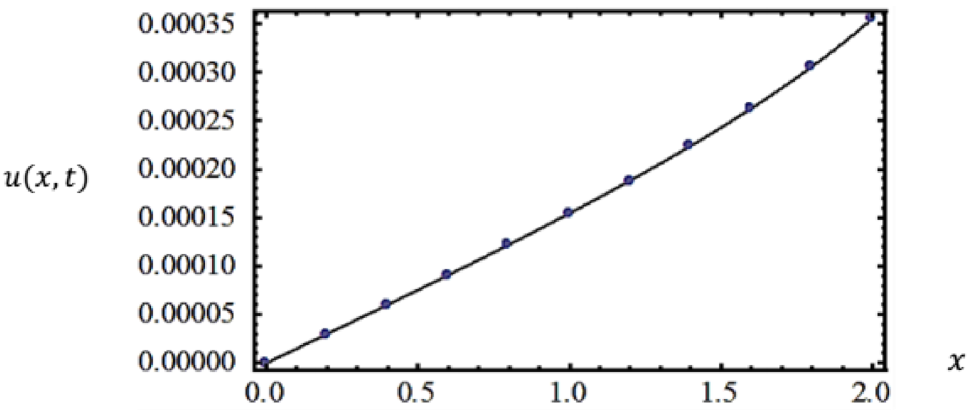


**Figure 5: Exact and Numerical solution of example 2 at *t=0.5***

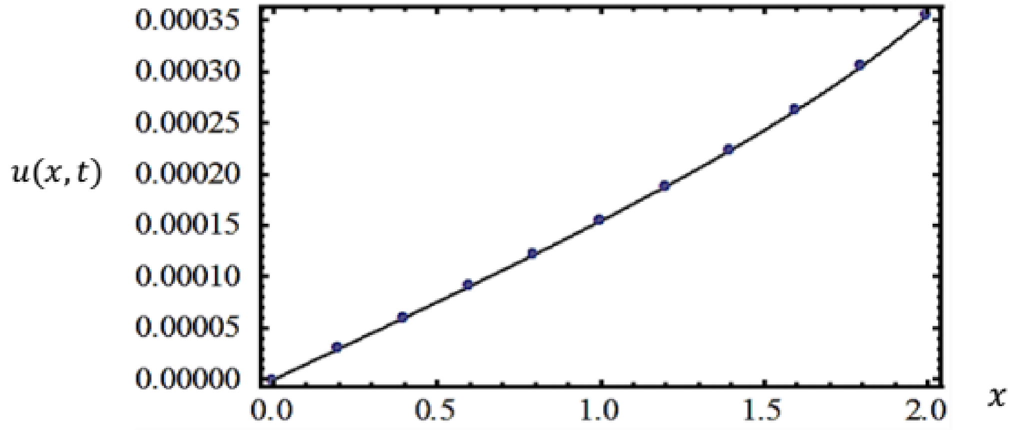


**Figure 6: Exact and Numerical solution of example 2 at *t=10***

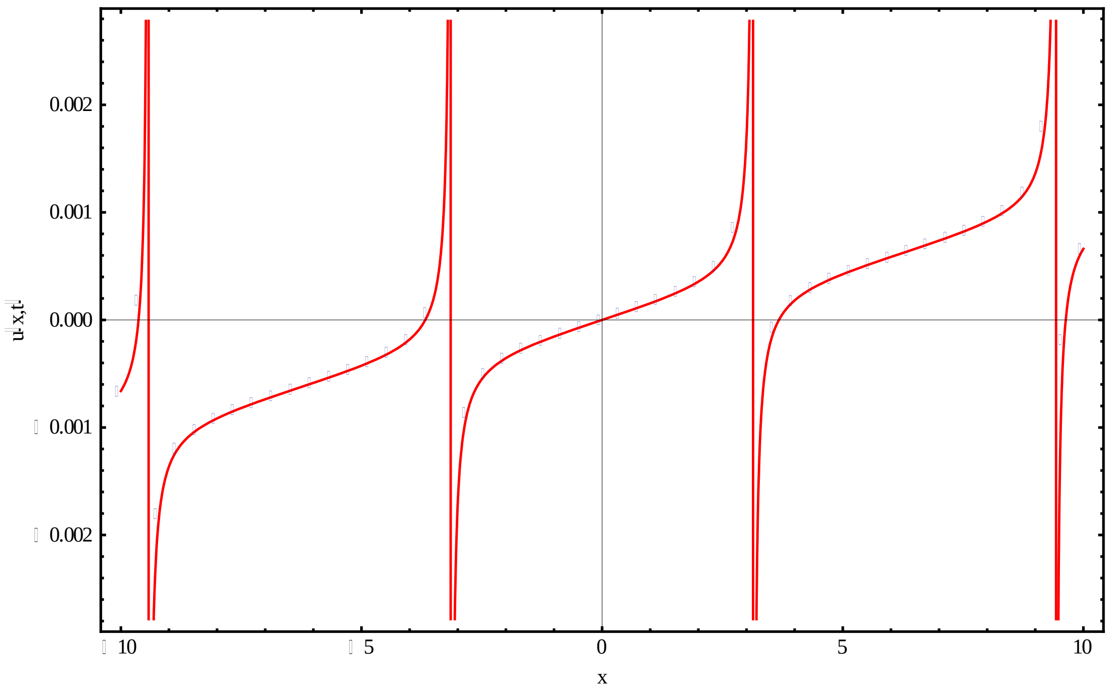


**Figure 7 : RTA solutions (square) and exact solutions (Solid) at times *t = 1* of Example 2**

**THE STABILITY ANALYSIS MODIFIED BURGER'S EQUATION**

Using Von Neuman method for the finite difference equation (8), assume a Fourier component for $u_{i,j}$ as defined in the form

$$u_{i,j} = \xi^j\, e^{Ikhi} = \xi^j e^{I\theta i} \qquad (16)$$

where $I = \sqrt{-1}$, $\xi^j$ is the amplitude at time level $k$ is the wave number and $h = \Delta x$.

And the nonlinear term $u^2 = M$ to linearize the nonlinear equation, then the stability condition of the finite difference equation (7) is $|\xi| \leq 1$

where

$$|\xi| = \left| \left( \frac{2k\, \epsilon_{i,L_1}\, \gamma}{h^2} \cos\theta + 1 - \frac{2k\, \epsilon_{i,L_1}\, \gamma}{h^2} \right) + I\, \frac{k\, \epsilon_{i,L_1}\, M}{h} \sin\theta \right| \leq 1$$

So the stability condition is:

$$\frac{2k\, \epsilon_{i,L_1}\, \gamma}{h^2} \leq 1 \qquad (17)$$

γ must be positive

## CONCLUSIONS

The Restrictive Taylor Approximation (RTA) is a type of finite difference approximation. The results in tables are close to exact solution. This proves that our restrictive method at some cases obtains the exact solution. The executive time of calculating of Restrictive Taylor Approximation is relatively very small. RTA gives the numerical solution which is very close to the exact solution if it is known at one level of time, for example at $t = k, i.e. u(x,t) = u(ih,k)\ \ i = 1(1)N$. Without knowing the exact solution at one level, we try to use an approximate, fast efficient and accurate method with suitable very small step sizes h and k, to get the needed almost exact solution at specific level, after which we continue the usual Restrictive Taylor process.

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